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The interconnection of sliding and rolling in the problem of the motion of a homogeneous sphere on a rough horizontal plane☆

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ABSTRACT

The motion of a homogeneous sphere on a rough horizontal plane when the angular velocities of the twisting and spinning of the sphere are equal to zero at the initial instant is considered. It is proved that, for any initial conditions, the angular velocity of the rolling of the sphere and the sliding velocity vanish after the same finite time. It is shown that the sliding and rolling are interconnected and, in particular, that the rolling of a sphere without sliding is impossible.

We consider the problem of the motion of a homogeneous sphere of mass m and radius a on a fixed rough horizontal plane. Unlike Zhuravlev's assumptions,[1,2] we shall assume that the interaction of the sphere with the plane is described by the friction model proposed by Karapetyan.[3,4], We will assume that $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is a right-handed, orthonormalized moving frame of reference such that the unit vector $\mathbf{e}_1$ is directed along the sliding velocity, $\mathbf{u}=u\mathbf{e}_1$, of the sphere (that is, along the velocity of the point of contact of the sphere with the support plane), the unit vector $\mathbf{e}_2$ is orthogonal to the sliding velocity $\mathbf{u}$ and, like $\mathbf{e}_1$, lies in the horizontal plane, and the unit vector $\mathbf{e}_3$ is directed along the ascending vertical, $\mathbf{u}_s=\mathbf{u}+[\boldsymbol{\omega}, a\mathbf{e}_3]$ is the velocity of the centre of mass of the sphere and $\boldsymbol{\omega}=\omega_1\mathbf{e}_1+\omega_2\mathbf{e}_2+\omega_3\mathbf{e}_3$ is the angular velocity of the sphere.

Assuming that the sphere moves without separation and using basic theorems of mechanics, we write out the equation of motion of the sphere on a rough horizontal plane

$$m\frac{d\mathbf{u}_S}{dt} = (N-mg)\mathbf{e}_3+\mathbf{F}, \quad \frac{2}{5}ma^2\frac{d\boldsymbol{\omega}}{dt} = [-a\mathbf{e}_3, \mathbf{F}]+\mathbf{M} \tag{1}$$

Here g is the gravitational acceleration, N is the normal pressure, $\mathbf{F}=F_1\mathbf{e}_1+F_2\mathbf{e}_2+F_3\mathbf{e}_3$ is the friction force acting on the sphere and applied at the point of contact of the sphere with the support plane and $\mathbf{M}=M_1\mathbf{e}_1+M_2\mathbf{e}_2+M_3\mathbf{e}_3$ is the principal moment of the friction forces about the point of contact of the sphere with the support plane. The first equation of (1) expresses the change in the momentum of the sphere and the second expresses the change in the angular momentum about the centre of mass of the sphere.

Taking account of the fact that the frame of reference $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ can rotate around a vertical and denoting its velocity by $\boldsymbol{\Omega}=\Omega\mathrm{e}_3$, we write out Eqs, (1) in projections onto the fixed axes associated with this frame of reference:

$$\dot{u}+a(\dot{\omega}_2+\omega_1\Omega) = f_1, \quad u\Omega-a(\dot{\omega}_1-\omega_2\Omega) = f_2, \quad N = mg$$
$$a(\dot{\omega}_1-\omega_2\Omega) = \frac{5f_2}{2}+\frac{5m_1}{2a}, \quad a(\dot{\omega}_2+\omega_1\Omega) = -\frac{5f_1}{2}+\frac{5m_2}{2a}, \quad a\dot{\omega}_3 = \frac{5m_3}{2a} \tag{2}$$

Here $f_i=F_i/m$, $m_j=M_j/m$ and differentiation with respect to time in the reference system associated with the frame of reference $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is denoted by a dot.

Suppose $\omega_1(0)=0$ and $\omega_3(0)=0$ at the initial instant. Then, within the limits of the model considered, [5], $f_2=m_1=m_3\equiv 0$. This means that the system of equations (2) admits of a solution of the form

$$\omega_1\equiv 0, \quad \omega_3\equiv 0, \quad \Omega\equiv 0, \quad u = u^0(t), \quad \omega_2 = \omega_2^0(t) \tag{3}$$

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The investigation of the dynamics of the sphere for the motions (3) reduces to an analysis of the system of equations

$$\dot{u} = \frac{7f_1^0}{2} - \frac{5m_2^0}{2a}, \quad \dot{w} = -\frac{5f_1^0}{2} + \frac{5m_2^0}{2a}(w = a\omega_2) \tag{4}$$

where [6]

$$f_1^0 = -kg\sigma + kg\sigma\mu^2\frac{7u^2 + \varepsilon^2 w^2}{70u^2} + O(\mu^3), \quad m_2^0 = -kga\sigma\varepsilon^2\mu^2\frac{w}{35u} + O(\mu^3) \tag{5}$$

$k > 0$ is the coefficient of friction and $\mu \in [0,1]$ and $\varepsilon \in [0,1]$ are the parameters of the friction model ($\mu \le \varepsilon$).

Theorem. The sliding velocity u and the angular rolling velocity of the sphere ω_2 vanish simultaneously. Furthermore, if, at the initial instant, the sliding velocity $u(0) = 0$ and $\omega_2(0)=\omega_0 \neq 0$, then a time interval $t\in(0, t_R)$ exists such that $u(t) \neq 0$ and $u(t)\omega_2(t) > 0$.

Proof. It has been proved [5] that the sphere is brought to a stop after a finite time t_R for any initial conditions. Using formula (5), we will prove that the sliding velocity u and the angular rolling velocity of the sphere ω_2 simultaneously vanish. To do this, we write the system of equations (4) in the form of the equation

$$\frac{du}{dw} = \frac{7f_1^0 - 5m_2^0/a}{-5f_1^0 + 5m_2^0/a}$$

and change from the variables u and w to the variables $z = u/w$ and $\tau=-\ln[w/w(0)]$.

As a result, we obtain

$$\frac{dz}{d\tau} = \frac{35(10-\mu^2)z^3 + (49(10-\mu^2) - 10\mu^2\varepsilon^2)z^2 - 15\mu^2\varepsilon^2 z - 7\mu^2\varepsilon^2}{5(7(10-\mu^2)z^2 - 2\mu^2\varepsilon^2 z - \mu^2\varepsilon^2)} \tag{6}$$

We now consider the function

$$f(z) = 5Az^3 + Bz^2 - 15\varepsilon^2\mu^2 z - 7\varepsilon^2\mu^2; \quad A = 7(10-\mu^2), \quad B = 7A - 10\varepsilon^2\mu^2 \tag{7}$$

It is well known that a third order polynomial has three real roots if and only if its discriminant is positive. The discriminant of polynomial (7) has the form

$$\Delta = 28\varepsilon^2\mu^2 B^3 + 225\varepsilon^4\mu^4 B^2 + 67\,500A\varepsilon^6\mu^6 + 4725\varepsilon^4\mu^4(7A - 20\varepsilon^2\mu^2)$$

Since $\mu \in [0, 1]$ and $\varepsilon \in [0, 1]$, we obtain that $A > 0$ and $7A-20\varepsilon^2\mu^2>0$. Consequently, $\Delta > 0$.

The function (7) can therefore be represented in the form

$$f(z) = 5A(z-\zeta_1)(z-\zeta_2)(z-\zeta_3), \quad \zeta_i \in \mathbf{R}, \quad i = 1, 2, 3 \tag{8}$$

This means that the system of equations (4) has a first integral (C is an arbitrary constant)

$$|u - \zeta_1 w|^{\gamma_1}|u - \zeta_2 w|^{\gamma_2}|u - \zeta_3 w|^{\gamma_3} = C|w|^{\gamma_4}, \quad \gamma_i \in \mathbf{R}, \quad i = 1, 2, 3, 4 \tag{9}$$

Hence, the sliding velocity u and the angular rolling velocity of the sphere ω_2 simultaneously vanish.

We will now show that rolling of the sphere without it sliding is impossible.

We will first prove the existence of the required time interval $t \in (0, t_R)$. We assume the opposite, that is, that the sliding velocity $u(t)\equiv 0$ during the whole motion. Then, until the angular rolling velocity of the sphere $\omega_2(t)$ vanishes, we have

$$f_1^0 = -\frac{kg}{10}\frac{\omega_2}{|\omega_2|}\mu^2 + O(\mu^3), \quad m_2^0 = -\frac{kga}{5}\frac{\omega_2}{|\omega_2|}\mu\varepsilon + O(\mu^3)$$

and the system of equations (4) takes the form

$$\mu = \frac{10\varepsilon}{7}, \quad a\dot{\omega}_2 = -\frac{kg}{4}\frac{\omega_2}{|\omega_2|}\mu(2\varepsilon - \mu)$$

Since $\mu \le \varepsilon$, the latter equations are incompatible.

We will now prove that

$$u(t)\omega_2(t) > 0 \quad \text{when} \quad t \in (0, t_R) \tag{10}$$

If $u(0)=0$, $\omega_2(0)=\omega_0 \neq 0$ at the initial instant, then, when $t \to 0+$, system of equations (4) takes the form

$$\dot{u} = \frac{kg}{20}\frac{\omega_0}{|\omega_0|}\mu(10\varepsilon - 7\mu), \quad a\dot{\omega}_2 = -\frac{kg}{4}\frac{\omega_0}{|\omega_0|}\mu(2\varepsilon - \mu) \tag{11}$$

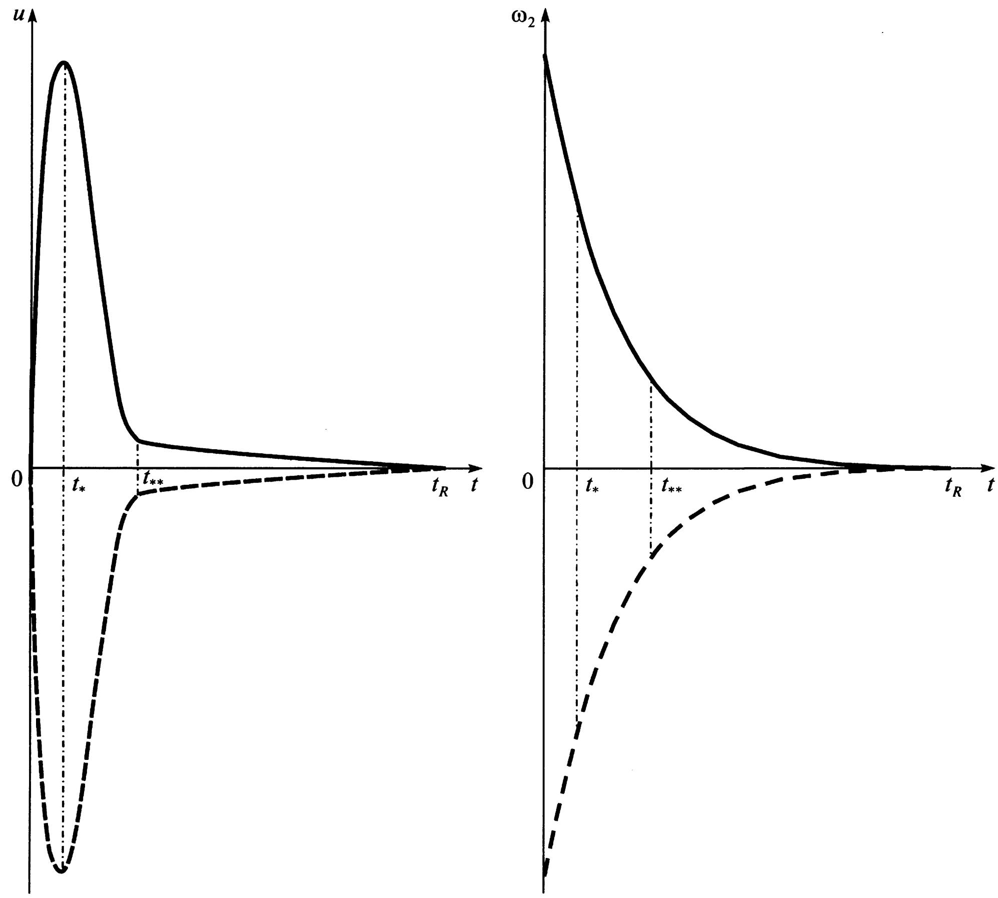

Fig. 1.

It follows from system (11) that, if the angular rolling velocity of the sphere ω_0 is positive (negative) at the initial instant, then the sliding velocity u increases (decreases) and the angular rolling velocity ω_2 decreases (increases) in a certain time interval $t \in (0, t_*) \subset (0, t_R)$. Allowing for the fact that these quantities vanish simultaneously, we conclude that inequality (10) holds.

Note that the results obtained analytically above correspond to the results of numerical integration of the equations of motion of the sphere (4). Numerical integration was carried out for $\varepsilon = 0.1$, $\mu = 0.001$ (these values of the parameters of the problem occur, for example, in the case of a billiard ball on a table with a glass covering), a coefficient of friction $k = 0.2$, a mass of the sphere $m = 15$ H 10^{-2} kg, a radius of the sphere a 29 H 10^{-3} m (a snooker ball) and initial conditions of the form $u(0)=0$, $\omega_2(0)=\omega_0 \neq 0$.

The results of the numerical calculations are presented in Fig. 1 in the form of the time dependence of the sliding velocity u (on the left) and the time dependence of the angular rolling velocity of the sphere ω_2 (on the right). The solid lines correspond to the case when $\omega_0 > 0$ and the dashed lines to the case when $\omega_0 < 0$.

It can be seen from these graphs that the overall time of motion of the sphere in this case can be subdivided into three intervals: in the first time interval $[0, t_*]$, the sliding velocity of the sphere "rapidly" increases (decreases) to its maximum (minimum) value u_* when $\omega_0>0$ ($\omega_0<0$). In the second time interval $[t_*, t_{**}]$, the sliding velocity u "rapidly" decreases (increases) to values of the order of μ and the angular rolling velocity ω_2 slowly decreases (increases) to the value ω_{**} when $\omega_0>0$ ($\omega_0<0$). In the third time interval $[t_{**}, t_R]$, both variables slowly decrease to zero.

Hence, the use of a new friction model in the study of the dynamics of a homogeneous sphere on a rough horizontal surface leads to results which differ from those obtained for the same problem within the limits of the Contensou-Zhuravlev model. It has been shown [1,7] that both the sliding and the rotation of the sphere cease after the same time, after which the sphere rolls uniformly along a fixed straight line. we have shown that the sliding and rolling of the sphere cease simultaneously, that is, it is impossible for the sphere to roll without sliding.

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